

Bounding Ornstein-Uhlenbeck Processes and Alikes

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Abstract

In this note we consider SDEs of the type $dX_t = [F(X_t) - AX_t]dt + DdW_t$ under the assumptions that A 's eigenvalues are all of positive real parts and $F(\cdot)$ has slower-than-linear growth rate. It is proved that $\overline{\lim}_{t \rightarrow \infty} \frac{\|X_t\|}{\sqrt{\log t}} = \sqrt{2\lambda_1}$ almost surely with λ_1 being the largest eigenvalue of the matrix $\Sigma := \int_0^\infty e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$; the discarded measure-zero set can be chosen independent of the initial values $X_0 = x$.

1 Introduction

It's well known that, for a given one-dimensional stationary Ornstein-Uhlenbeck (OU for short) process $X = \{X_t : t \geq 0\}$ there exist $\lambda, \sigma > 0, \mu \in \mathbb{R}$ and a standard Brownian Motion (BM for short) $B(\cdot)$ such that X has the same distribution as $\{\sigma \cdot e^{-\frac{\lambda t}{2}} \cdot B(e^{\lambda t}) + \mu : t \geq 0\}$. Therefore the law of iterated logarithm for BM (see, e.g., [1]) leads us to the conclusion $X_t = O(\sqrt{\log t})$ almost surely. In this note we investigate what bounds can we achieve for higher dimensional OU processes $X = \{X_t : t \geq 0\}$ and alike which may be modeled by the following SDE (of dimension $d \geq 2$)

$$dX_t = [F(X_t) - AX_t]dt + D dW_t, \quad (1.1)$$

where D is a constant d -by- d matrix. And we always assume the following conditions:

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(C1) All the eigenvalues of the d -by- d matrix A have positive real parts;

(C2) $F(x) = o(\|x\|)$ (as $\|x\| \rightarrow \infty$) is a continuous \mathbb{R}^d -valued function. Here $\|\cdot\|$ denotes the standard Euclidean norm.

Our main result can be stated as the following.

Theorem 1 *The solution to (1.1) always satisfies*

$$\overline{\lim}_{t \rightarrow \infty} \frac{\|X_t\|}{\sqrt{\log t}} = \sqrt{2\lambda_1} \text{ almost surely,} \quad (1.2)$$

where λ_1 is the largest eigenvalue of the matrix $\Sigma := \int_0^\infty e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$. Here the discarded measure-zero set can be chosen independent of the initial values $X_0 = x$.

Such result seems to be new in literature as to our knowledge and deserves a publication somewhere. The proof of the main theorem, based mainly on the well-known fact mentioned at the beginning of the introduction and on elemental linear algebra, is presented in Sect. 2 and Sect. 3; the calculation of the precise limit value in (1.2) is based mainly on [2], see Sect. 2.

2 OU Processes Case: $F = 0$

In this part, we consider the simpler case of $F = 0$, i.e., the follow model

$$dX_t = -AX_t dt + D dW_t. \quad (2.1)$$

Clearly the solution satisfies the follow formula

$$X_t = e^{-tA} X_0 + \int_0^t e^{-(t-s)A} D dW_s. \quad (2.2)$$

When $X_0 \sim N(0, \Sigma)$ with $\Sigma := \int_0^\infty e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$, $\{X_t : t \geq 0\}$ is a stationary Markov process.

Throughout this section, we will use B in denoting one dimensional standard BM and write W for higher dimensional standard BM.

As we have addressed in the introduction, any one dimensional stationary OU process is of growth rate $O(\sqrt{\log t})$. This result can be restated as the following lemma, whose proof is omitted.

Lemma 2 *For any $\lambda > 0$, almost surely we have*

$$\int_0^t e^{-\lambda(t-s)} dB_s = O(\sqrt{\log t}).$$

Based on the above lemma, we would prove the following three lemmas.

Lemma 3 *For $\lambda > 0$ and any $k \in \mathbb{N}$, almost surely we have*

$$\int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} dB_s = O(\sqrt{\log t}).$$

Lemma 4 *For $\lambda > 0, \mu \neq 0$, almost surely we have*

$$\begin{aligned} \int_0^t e^{-\lambda(t-s)} \cdot \cos \mu(t-s) dB_s &= O(\sqrt{\log t}), \\ \int_0^t e^{-\lambda(t-s)} \cdot \sin \mu(t-s) dB_s &= O(\sqrt{\log t}). \end{aligned}$$

Lemma 5 *For $\lambda > 0, \mu \neq 0$ and any $k \in \mathbb{N}$, almost surely we have*

$$\begin{aligned} \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} \cdot \cos \mu(t-s) dB_s &= O(\sqrt{\log t}), \\ \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} \cdot \sin \mu(t-s) dB_s &= O(\sqrt{\log t}). \end{aligned}$$

Proof of Lemma 3. Put $Y_t := \int_0^t e^{-\lambda(t-s)} dB_s$ and

$$L(t) := \sup_{u \in [0, t]} |Y_u|, \quad I_t^{(k)} := \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} dB_s.$$

Clearly

$$I_t^{(1)} = \int_0^t e^{-\lambda(t-u)} Y_u du, \quad I_t^{(k+1)} = \int_0^t e^{-\lambda(t-u)} I_u^{(k)} du, \quad k \geq 1.$$

And

$$|I_t^{(1)}| \leq \int_0^t e^{-\lambda(t-u)} |Y_u| du \leq \int_0^t e^{-\lambda(t-u)} L(u) du \leq L(t)/\lambda.$$

Lemma 2 tells us $L(t) = O(\sqrt{\log t})$. Hence $I_t^{(1)} = O(\sqrt{\log t})$. Inductively $I_t^{(k)} = O(\sqrt{\log t})$

for all $k \in \mathbb{N}$. □

Proof of Lemma 4. For any $\theta \in \mathbb{R}$, we write

$$R_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

These are rotations which preserve the distance induced by the standard norm $\|\cdot\|$ on \mathbb{R}^2 .

Consider the following diffusion process

$$X_t := \int_0^t e^{-\lambda(t-s)} \cdot R_{-\mu(t-s)} dW_s,$$

where $W = (W^1, W^2)^T$ is a 2-dimensional standard BM. Define

$$\widetilde{W}_t := \int_0^t R_{\mu s} dW_s.$$

It is easy to see that \widetilde{W} is still a 2-dimensional standard BM. And

$$X_t = \int_0^t e^{-(t-s)} \cdot R_{-\mu t} d\widetilde{W}_s.$$

Now in view of Lemma 2 it is clear that

$$\|X_t\| = \left\| \int_0^t e^{-(t-s)} d\widetilde{W}_s \right\| = O(\sqrt{\log t}).$$

Thus

$$\int_0^t e^{-\lambda(t-s)} \cdot \left[\cos \mu(t-s) dW_s^1 - \sin \mu(t-s) dW_s^2 \right] = O(\sqrt{\log t}).$$

Similarly,

$$\int_0^t e^{-\lambda(t-s)} \cdot \left[\cos \mu(t-s) dW_s^1 + \sin \mu(t-s) dW_s^2 \right] = O(\sqrt{\log t}).$$

The lemma follows from the above equations. □

Proof of Lemma 5. Now for any $k \geq 1$ (fixed), consider

$$X_t := \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} \cdot R_{-\mu(t-s)} dW_s,$$

where R is introduced in the proof of Lemma 4. It is easy to see that

$$\|X_t\| = \left\| \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} d\widetilde{W}_s \right\|,$$

where \widetilde{W} is also introduced in the proof of Lemma 4. Now Lemma 3 tells us $\|X_t\| = O(\sqrt{\log t})$ and the rest proof follows smoothly as in the proof of Lemma 4. \square

From the above four lemmas we easily prove the bound $O(\sqrt{\log t})$ for the solutions to (2.1) via exploiting the standard Jordan form of A (and hence of $e^{-(t-s)A}$) in formula (2.2). The fact that the $\overline{\lim}$ in (1.2) is constant almost surely follows from the ergodic property of the stationary OU process.

Now we calculate the $\overline{\lim}$ in (1.2) explicitly via [2]: Without loss of generality, assume Σ to be invertible. Take $V(x) = \frac{1}{2}x^T \Sigma^{-1}x$, the result in [2] tells us that $\overline{\lim}_{t \rightarrow \infty} \frac{V(X_t)}{\sqrt{\log t}} \leq 1$ almost surely, which implies

$$\overline{\lim}_{t \rightarrow \infty} \frac{\|X_t\|}{\sqrt{\log t}} = c \leq \sqrt{2\lambda_1} \text{ almost surely.}$$

Now let α be a unit eigen-vector of Σ corresponding to λ_1 . It is easy to see that $Y := \{Y_n := \alpha^T X_n / \sqrt{\lambda_1} : n \geq 0\}$ is a stationary Gaussian process with steady distribution $N(0, 1)$; This process inherits the exponential mixing property from X . A standard result says that for i.i.d. standard normal random variables $\{Z_n : n \geq 0\}$, we always have $\overline{\lim}_{n \rightarrow \infty} \frac{|Z_n|}{\sqrt{2 \log n}} = 1$ almost surely. With a tedious but routine effort (which we omit the details here), it is not hard to see that we still have $\overline{\lim}_{n \rightarrow \infty} \frac{|Y_n|}{\sqrt{2 \log n}} = 1$ almost surely for the new process Y . Therefore

$$c = \overline{\lim}_{t \rightarrow \infty} \frac{\|X_t\|}{\sqrt{\log t}} \geq \overline{\lim}_{n \rightarrow \infty} \frac{\|X_n\|}{\sqrt{\log n}} = \sqrt{2\lambda_1}.$$

Hence $c = \sqrt{2\lambda_1}$. And (1.2) follows.

3 General Case: $F \neq 0$

Now we consider the general case with $F \neq 0$. As is known, the solution to (1.1) satisfies

$$X_t = e^{-tA}X_0 + \int_0^t e^{-(t-s)A}F(X_s)ds + \int_0^t e^{-(t-s)A}D dW_s. \quad (3.1)$$

Define

$$L(t) = \sup_{u \in [0, t]} \left\| \int_0^u e^{-(u-s)A}D dW_s \right\|.$$

Clearly $L(t) = O(\sqrt{\log t})$.

Since A satisfies condition **(C1)**, there exist $\lambda_0 > 0$ and $K > 0$ such that

$$\|e^{-tA}\| \leq K \cdot e^{-\lambda_0 t}, \quad \forall t \geq 0. \quad (3.2)$$

Fix an arbitrarily small $\varepsilon > 0$ (with $\varepsilon < \frac{\lambda_0}{K}$), by assumption **(C2)** there exists $C = C(\varepsilon) > 0$ such that

$$\|F(x)\| \leq C + \varepsilon\|x\|, \quad \forall x \in \mathbb{R}^d. \quad (3.3)$$

In view of (3.1) we have

$$\|X_t\| \leq Ke^{-\lambda_0 t}\|X_0\| + \int_0^t Ke^{-\lambda_0(t-s)}(C + \varepsilon\|X_s\|)ds + L(t), \quad \forall t \geq 0.$$

Define

$$f(t) := K\|X_0\| + L(t) + KC/\lambda_0, \quad \varepsilon_0 := K\varepsilon, \quad u(t) := \|X_t\|.$$

Then $u(\cdot)$ can be regarded as a continuous positive function (almost surely) which satisfies the following inequality

$$u(t) \leq f(t) + \varepsilon_0 \int_0^t e^{-\lambda_0(t-s)}u(s)ds, \quad \forall t \geq 0. \quad (3.4)$$

Put $\varphi(t) := \int_0^t e^{\lambda_0 s}u(s)ds$, we have

$$\frac{d\varphi}{dt}(t) \leq f(t) \cdot e^{\lambda_0 t} + \varepsilon_0 \varphi(t), \quad t \geq 0$$

which implies (where $\lambda := \lambda_0 - \varepsilon_0$)

$$\varphi(t) \leq e^{\varepsilon_0 t} \cdot \int_0^t f(s) \cdot e^{\lambda s} ds, \quad t \geq 0.$$

Thus, noting (3.4) and the monotonicity of f , we have

$$\begin{aligned} u(t) &\leq f(t) + \varepsilon_0 e^{-\lambda_0 t} \cdot \varphi(t) \leq f(t) + \varepsilon_0 \int_0^t f(s) e^{-\lambda(t-s)} ds \\ &\leq \left[1 + \frac{\varepsilon_0}{\lambda_0 - \varepsilon_0}\right] \cdot f(t) = \frac{\lambda_0}{\lambda_0 - \varepsilon_0} \cdot f(t). \end{aligned}$$

This implies that the solution X_t has almost the same growth rate as $\int_0^t e^{-(t-s)A} D dW_s$.

Specifically we always have $\|X_t\| = O(\sqrt{\log t})$ almost surely. Clearly the limit value in (1.2) is coincident with that of the OU case (i.e., the case $F = 0$).

Acknowledgements The author thanks Prof. Jiangang Ying for helpful discussions.

This work is partially supported by NSFC (No. 10701026 and No. 11271077) and the Laboratory of Mathematics for Nonlinear Science, Fudan University.

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